

Index theorem, spin Chern Simons theory and fractional magnetoelectric effect in strongly correlated topological insulators

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Making use of index theorem and spin Chern Simons theory, we construct an effective topological field theory of strongly correlated topological insulators coupling to a nonabelian gauge field $SU(N)$ with an interaction constant g in the absence of the time-reversal symmetry breaking. If N and g allow us to define a t'Hooft parameter λ of effective coupling as $\lambda = Ng^2$, then our construction leads to the fractional quantum Hall effect on the surface with Hall conductance $\sigma_H^s = \frac{1}{4\lambda} \frac{e^2}{h}$. For the magnetoelectric response described by a bulk axion angle θ , we propose that the fractional magnetoelectric effect can be realized in gapped time reversal invariant topological insulators of strongly correlated bosons or fermions with an effective axion angle $\theta_{eff} = \frac{\pi}{2\lambda}$ if they can have fractional excitations and degenerate ground states on topologically nontrivial and oriented spaces. Provided that an effective charge is given by $e_{eff} = \frac{e}{\sqrt{2\lambda}}$, it is shown that $\sigma_H^s = \frac{e_{eff}^2}{2h}$, resulting in a surface Hall conductance of gapless fermions with e_{eff} and a pure axion angle $\theta = \pi$.

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Recently topological insulators (TIs) have received a great deal of attention [1–3]. The theory of TI has been developed along a few different directions. As one of developments, topological band theory has been described in terms of Z_2 topological invariants for noninteracting band insulators in $(3 + 1)$ dimensions (Ds) when the time-reversal symmetry (TRS) is not broken [4–7]. TI theory has also been proposed in HgTe quantum wells [8] in the presence of spin-orbit interactions. Furthermore, there have been many experimental observations supporting the existence of nontrivial topological surface states in numerous materials [1–3, 9, 10].

As another direction of TI theory, topological field theories (TFTs) have been suggested in the low energy limit of TIs [11, 12]. In particular, analogous to the coupling of an axion particle to ordinary electric and magnetic fields, the partition function of TFT is expressed by

$$\mathcal{Z}(F) = C \exp(iK_\theta \frac{e^2}{4\pi} \int_{M_4} F \wedge F), \quad (1)$$

where C is a constant, and F is the electromagnetic field strength [12, 13]. Here K_θ is a bulk magnetoelectric polarization written by $K_\theta = \frac{\theta}{2\pi}$. Under shifts of θ by multiples of 2π , the partition function and all physical quantities are invariant on the periodic boundary condition. But the θ term has a problem dangerous to the TRS. It follows that trivial insulators have $\theta = 0 \bmod 2\pi$ whereas noninteracting TIs take $\pi \bmod 2\pi$ in the values allowed by a time (T) operation. Without breaking the TRS, the current theoretical challenge is how to extend noninteracting TIs to the strongly correlated electron systems of TIs [12, 14, 15]. It has more recently been proposed that there is a possibility of T-invariant fractional TIs for

fermions and bosons in correlated systems [16–18]. More interesting issue is to formulate the general TI theory for strongly correlated systems that cannot be smoothly connected to any band insulator.

In this article, using flux quantization, index theorem and spin Chern Simons (CS) theory, we present a fractional magnetoelectric effect of strongly correlated TIs coupling to a non-abelian gauge field $SU(N)$ with an interaction constant g on an easy and simple effective field theory (FT) under the TRS and topological gauge invariance. The low energy FT can provide the interacting $SU(N)$ gauge theory in TIs unless the TRS is spontaneously broken. If a t'Hooft parameter $\lambda = Ng^2$ is defined as an effective interaction constant, then our construction can lead to the fractional quantum Hall effect (FQHE) on the surface with Hall conductance $\sigma_H^s = \frac{1}{4\lambda} \frac{e^2}{h}$. In the viewpoint of the magnetoelectric response for the interacting TIs, the fractional axion angle allows us to result in the fractional magnetoelectric polarization $K_\theta = \frac{1}{4\lambda}$. It is proposed that the fractional magnetoelectric effect can be realized in gapped time reversal invariant topological insulators of strongly correlated bosons or fermions with an effective axion angle $\theta_{eff} = \frac{\pi}{2\lambda}$ if they can have fractional excitations and degenerate ground states on topologically nontrivial and oriented spaces.

On the other hand, provided that an effective charge can be expressed in terms of $e_{eff} = \frac{e}{\sqrt{2\lambda}}$, the surface Hall conductance is regarded as a conductivity $\sigma_H^s = \frac{e_{eff}^2}{2h}$ for a single Dirac cone of gapless fermions with fractional charge e_{eff} and a pure axion angle $\theta = \pi$. Recently, angular photoemission spectroscopy experiment has showed that there is a topological phase transition from a trivial

insulator into a topological surface state by displaying the emergence of spin vortex with fractional charge $\pm\frac{e}{2}$ and $\theta = \pi$ in a tunable topological insulator $\text{BiTl}(\text{S}_{1-\delta}\text{Se}_\delta)$ [19]. Theoretically it has been suggested that there can be a quantized vortex of fractional charge $\pm\frac{e}{2}$ and an odd number of gapless Dirac fermions at the surface of a strong TI [20]. As a special viewpoint of these experimental and theoretical studies, when $\lambda = 2$, our theoretical construction shows that the TIs can have fractionalized charge $\frac{e}{2}$ topological objects with a bulk gap and string-like vortex excitations. These topological excitations can be described in terms of a deconfined Z_2 gauge theory in (3+1)D.

The article is organized as followings. We will review parton models of effective FT for correlated TIs in Section II. In Section III, Dirac quantization conditions will be illuminated in the cases of complex spinor fields on a Riemannian manifold with spin structures and nonempty spin boundary. For complex spinor fields on the spin surface, the Chern-Simon (CS) theory and Atiyah-Patodi-Singer (APS) index theorem will be exploited to investigate the effective QFT for the interacting nonabelian gauge fields of correlated TIs in Section IV. In V, we will describe a fractional surface Hall conductance of correlated TIs and will use the APS theorem to explain a general form for the Hall conductance and magnetoelectric effects on a Riemann spin surface with a genus in the effective QFT. And finally we will come to summary and conclusion for our results in Section VI.

II. REVIEW OF PARTON MODELS FOR EFFECTIVE FIELD THEORY

Let us review parton models of effective FT in interacting topological insulators. Then for a more systematical approach of the interacting TIs, we take into account the projective construction of a correlated electron system for building the effective FT on a Riemann surface emerged from a (3 + 1)D Riemannian manifold with spin structures. An electron is decomposed into N different fractionally charged and fermionic partons [21, 22]. It is without loss of generality that the partons generate a ground state of topological phases. When the partons get together to create the physical real electrons, a new ground state of topological phase can appear from their recombination. Provided that the partons are recombined together to represent the physical electrons, we can construct an interacting many-body wavefunction as a new topological state of electrons emerged from (3 + 1)D.

One can generalize this construction to N_f different flavors of charged fermion partons, with N_f^c partons for each flavors $f = 1, \dots, N_f$. The electron is fractionalized into N_f different flavors of fractionally charged fermion partons, with N_f^c partons for each flavors. Under these

decompositions, we should obey two crucial constraints. First, because the electron preserves the fermion statistics, the total number of partons per electron has to be odd such that

$$N_1^c + \dots + N_{N_f}^c = \text{odd}. \quad (2)$$

The second constraint is that the total charge of the partons should sum up to the electron charge e such as

$$N_1^c q_1 + \dots + N_{N_f}^c q_{N_f} = e \quad (3)$$

when $q_f < e$ is the fractional charge for partons of flavor f . The total electron wavefunction is expressed by a product of parton ground state wavefunctions [23]

$$\prod_{f=1}^{N_f} \Psi_{N_f^c}(\{\mathbf{r}_i, \mathbf{s}_i\}) = \prod_{f=1}^{N_f} [\Psi_f(\{\mathbf{r}_i, \mathbf{s}_i\})]^{N_f^c}. \quad (4)$$

Here $\Psi_f(\{\mathbf{r}_i, \mathbf{s}_i\})$ stands for the parton ground state wavefunction given by a Slater determinant which describe the ground state of a noninteracting TI Hamiltonian, and $\{\mathbf{r}_i, \mathbf{s}_i\}, i = 1, \dots, N$, the position and spin coordinates of the partons.

For a more specific approach for strongly correlated TI on a lattice of $SU(N)$ electrons, the Hamiltonian is expressed by

$$H = \sum_{ij} \{C_{i\alpha}^\dagger h_{ij}^{\alpha\beta} e^{ieA_{ij}} C_{j\beta} + H.C.\} + H_{int}(C^\dagger, C), \quad (5)$$

where i, j stands for site indices, α, β denotes internal degrees of freedom, h_{ij} indicates the Hamiltonian matrix, $A_{ij} = \int_{\mathbf{r}_i}^{\mathbf{r}_j} d\mathbf{r} \cdot \mathbf{A}$ with the $U(1)$ electromagnetic vector potential \mathbf{A} . H_{int} denotes an interaction Hamiltonian between electrons. $C_{i\alpha}$ is the electron operator that is decomposed into [23]

$$C_{i\alpha} = \prod_{f=1}^{N_f} \psi_{1\alpha}^f(\mathbf{r}_i) \cdots \psi_{N_f^c\alpha}^f(\mathbf{r}_i) \quad (6)$$

when satisfying the constraint rules of Eqs. (2) and (3).

It is well known that the quark operators can act on a Hilbert space larger than the physical electron one. The unphysical states which cannot become invariant under unitary transformations, should be removed from the quark Hilbert space. When the electron operators become preserved, those transformations provide the $SU(N_f^c)$ for quarks in the N_f^c representation of each flavor $f = 1, \dots, N_f$. Thus it follows that the projection can be taken onto the electron Hilbert space implemented by coupling minimally the quark to a $SU(N_f^c)$ gauge field a_μ with an interacting constant g . This gives rise to observation only of $SU(N_f^c)$ excitations in its low-energy spectrum. Quarks of a given flavor can be symmetric or antisymmetric in their N_f^c odd color indices with the constraints of Eq. (2) for $f = 1, \dots, N_f$.

III. DIRAC QUANTIZATION CONDITION OF SPINOR FIELDS

Now we begin with the simplest case of $N_f = 1$ and N_1^c odd. Dirac quantization of flux can have a nice way of topological gauge invariance in the representation of complex spinor fields of antisymmetric N_1^c partons with N_1^c odd for a composite electron in the interacting TI. Let us consider the Dirac quantization of fermions represented by complex spinor fields through a two-cycle Σ in the sense of antisymmetric N_1^c partons for a composite electron with the odd-number constraint of N_1^c on M_4 . Then it follows that we don't necessarily require a real (neutral) spinor field on M_4 . The spinor fields can have a natural connection to a possible obstruction of spin which can be described in terms of the second Stiefel-Whitney class ω as an element of $\mathbb{H}^2(M_4, \mathbb{Z}_2)$ which is called the second cohomology group with a coefficient \mathbb{Z}_2 over M_4 . In order to exist neutral spinors on M_4 , it is shown that its vanishing, i.e., mod 2, is required as a necessary and sufficient condition for them. On the condition that ω is well defined to mod 2, M_4 has a *spin* structure.

Assume that there are complex spinor fields on M_4 . Then they provide a deep insight for Dirac's quantization condition. Under the Dirac quantization of flux, the antisymmetric parton wavefunctions can be represented by $SU(N_1^c)$ spinors on M_4 which is covered by a finite number of neighborhoods U_i for $i = 1, 2, \dots, L$. In each neighborhood, more structures have to be taken into account on the representation of internal symmetries. For more structure, in addition to the $U(1)$ connection or gauge potential, we should consider an oriented frame of vierbein V_i for $i = 1, 2, \dots, L$, and complex spinor fields of antisymmetric partons $\{\Psi_{1i}\}^{N_1^c}$ with N_1^c only odd. These symmetries can be dependent on choices made in the neighborhood U_i . As choices of degrees of freedom, there can exist local $U(1)$ gauge transformations χ

$$\{\Psi_{1i}\}^{N_1^c} \rightarrow \{e^{iq_1 \chi_i} \Psi_{1i}\}^{N_1^c}, \quad A_i \rightarrow A_i + d\chi_i, \quad (7)$$

for $i = 1, 2, \dots, L$, and $SU(N_1^c)$ gauge transformations λ

$$V_i \rightarrow R V_i, \quad a_i \rightarrow a_i + d\lambda_i, \\ \{\Psi_{1i}\}^{N_1^c} \rightarrow \{S(R) e^{ig\lambda_i} \Psi_{1i}\}^{N_1^c}, \quad (8)$$

where $R \in SO(4)$ and for $i = 1, 2, \dots, L$. Here q_1 and g have indicated an electric charge of one flavor and an interaction constant of $SU(N_1^c)$, respectively. It is easily seen that there can be a sign ambiguity from the lift of $R \rightarrow \pm S(R)$ since the quotient of the spin group $Spin(4)$ by \mathbb{Z}_2 is isomorphic to $SO(4)$. In the sense of a double overlap on two contiguous neighborhoods, $U_i \cap U_j \neq \emptyset$, one must take transition functions associated with transformation groups

$$A_i \rightarrow A_j + d\chi_{ij}, \quad a_i \rightarrow a_j + d\lambda_{ij}, \quad V_i \rightarrow R_{ij} V_j, \\ \{\Psi_{1i}\}^{N_1^c} \rightarrow \{S(R_{ij}) e^{iq_1 \chi_{ij}} e^{ig\lambda_{ij}} \Psi_{1j}\}^{N_1^c}, \quad (9)$$

for $i = 1, 2, \dots, L$. There is without loss of generality to assume that

$$R_{ij} = (R_{ji})^{-1}, \quad \chi_{ij} = (\chi_{ji})^{-1}, \\ \lambda_{ij} = (\lambda_{ji})^{-1}, \quad S(R_{ij}) = (S(R_{ji}))^{-1}, \quad (10)$$

for $i, j = 1, 2, \dots, L$. On a triple overlap region, $U_i \cap U_j \cap U_k \neq \emptyset, \forall i, j, k = 1, 2, \dots, L$, which is supposed to be contractible, one can have consistency conditions

$$R_{ij} R_{jk} R_{ki} = I, \quad S(ijk) \equiv S(R_{ij}) S(R_{jk}) S(R_{ki}) = \pm I. \quad (11)$$

It follows that the above equations have identity elements of $SO(4)$ and $Spin(4)$. This returns the frame V_i to itself. Consequently, under $U(1) \times SU(N_1^c)$ gauge transformations and in the spinor representations of antisymmetric N_1^c partons with N_1^c odd, one can express

$$\{\Psi_{1i}\}^{N_1^c} = \{e^{iq_1(\chi_{ij} + \chi_{jk} + \chi_{ki})} e^{ig(\lambda_{ij} + \lambda_{jk} + \lambda_{ki})} \\ S(R_{ij}) S(R_{jk}) S(R_{ki}) \Psi_{1i}\}^{N_1^c}, \quad (12)$$

for $i, j, k = 1, 2, \dots, L$. In the above expression, the product of the three matrices S can play an essential role, and so $S(ijk)$ can indicate the product as a following

$$S(ijk) \equiv S(R_{ij}) S(R_{jk}) S(R_{ki}) = \pm I, \quad (13)$$

for $i, j, k = 1, 2, \dots, L$. Up to a sign, a crucial point, is to take a lift from $SO(4)$ to $Spin(4)$ in the right hand side of Eq. (13). One cannot decide the sign when it is dependent on the choices made in the transformation groups of Eq. (13). In the sense of different overlaps, the signs of Eq. (13) can not be totally independent. Thus the spinor consistency condition leads one to obtain

$$e^{iq_1 C_{ijk}} e^{ig D_{ijk}} = S(ijk), \quad (14)$$

in the representation of the complex antisymmetric wavefunctions with g in addition to the symmetric scalar ones with q_1 in terms of C_{ijk} . In Eq. (14), D_{ijk} satisfies the self-consistency relation

$$D_{ijk} \equiv \lambda_{ij} + \lambda_{jk} + \lambda_{ki} \in \frac{\mathbb{Z}}{g N_1^c} \quad (15)$$

for $i, j, k = 1, 2, \dots, L$. Therefore, under the flux of $U(1) \times SU(N_1^c)$ gauge theory through a two-cycle Σ , the Dirac quantization of complex spinor fields produce

$$\exp(2\pi i \int_{\Sigma} (q_1 F + g G)) = (-1)^{\omega(\Sigma)}, \quad (16)$$

where G is the $SU(N_1^c)$ field strength, and $q_1 = \frac{e}{N_1^c}$ with N_1^c odd while F is the $U(1)$. Here the sign is determined by the finite product over triple overlaps

$$(-1)^{\omega(\Sigma)} = \prod_{U_i \cap U_j \cap U_k \cap \Sigma \neq \emptyset} S(ijk). \quad (17)$$

Equation (16) is regarded as the preliminary expression for the Dirac quantization condition of fermions in the complex spinor representations of antisymmetric parton wavefunctions. The crucial idea is how $\omega(\Sigma)$ in question should be independent of the choices made on the covering of two-cycle Σ by neighborhoods. Without loss of generality, the preliminary result proves a natural connection for the Dirac quantization of flux certainly independent of those choices. Likely, Σ can not be changed under transformation of the two-cycle by a homologous one $\Sigma \rightarrow \Sigma' + \partial\Lambda$. By all these results, it is shown that Σ can be closely associated with the Stiefel-Whitney two-cocycle ω over Z_2 .

In general, let us extend a base Σ to a set of bases $\Sigma_1, \Sigma_2, \dots, \Sigma_L$ of the integer lattice. Suppose that $\omega_i = \omega(\Sigma_i), \forall i = 1, 2, \dots, L$. Then the quantization condition is extended to get shifted into [24–26]

$$\int_{\Sigma_i} \left(\frac{e}{N_1^c} F + gG \right) - \frac{1}{2} \omega_i, \forall i = 1, 2, \dots, L. \quad (18)$$

Therefore there can be possible on fractional values of flux through a two-cocycle Σ in the total $U(1) \times SU(N_1^c)$ gauge theory of quark fields described by antisymmetric N_1^c partons with N_1^c odd. One should still identify the number ω_i with the self-intersection one of Σ_i mod 2, $\forall i = 1, 2, \dots, L$.

Finally we take into account the flux through Σ_i with nontrivial boundary $\partial\Sigma_i, \forall i = 1, 2, \dots, L$. In the representation of complex spinor fields for electrons formed by antisymmetric partons, the spinor field consistency Eq. (18) gives rise to the boundary Dirac quantization condition

$$\begin{aligned} & \exp(2\pi i \int_{\Sigma} (q_1 F + gG)) \\ &= \exp(2\pi i \oint_{\partial\Sigma} (q_1 A + ga)) \prod_{U_i \cap U_j \cap U_k \cap \Sigma \neq \emptyset} S(ijk). \end{aligned} \quad (19)$$

The crucial problem is that the two factors within Eq. (19) can become dependent on the choice of neighborhoods although their choices are independent of the right hand side. It is argued that when adding a neighborhood to the interior of Σ , it cannot give an effect on the second factor and cannot unambiguously become affected to the first factor. But the more serious problem arises up provided that we can take an addition of a neighborhood to the covering of the boundaries $\partial\Sigma_i, \forall i = 1, 2, \dots, L$. This fact forces us to change the sign of both factors. Thus we cannot deal with the two factors individually as the intrinsic properties of given manifolds. In order to resolve this problem, if we could have a square of Eq. (19), the issue of sign can be well defined due to a double covering.

However, even if we cannot have any better idea for understanding Stokes' theorem in the current context, there can very useful if the boundary of Σ_i has a two cycle that can be represented as a double covering, say, $\partial\Sigma_i =$

$2\gamma_i, \forall i = 1, 2, \dots, L$. As a matter of fact, this boundary quantization scheme is mathematically a mapping to lift the $U(1) \times SU(N_1^c)$ gauge bundle to a total covering space that combines the gauge bundle manifold into the basis one. On the double covering condition, the boundary Dirac quantization gives rise to a form [24, 25]

$$\begin{aligned} & \exp(2\pi i \int_{\Sigma} (q_1 F + gG)) \\ &= (-1)^{\omega(\Sigma)} \exp(2\pi i \oint_{\partial\Sigma} 2(q_1 A + ga)). \end{aligned} \quad (20)$$

Under the fact that the two factors are independent of choices of neighborhoods, they can be well defined since the second factor is well-behaved in the case of inserting the number 2 in the exponent of equation (20). It is claimed that $\omega(\Sigma)$ becomes well-defined mod 2 on the Σ with the boundary quantization condition as a double covering map.

Now we apply the boundary quantization scheme of double covering to the Dirac flux quantization of $U(1) \times SU(N_1^c)$ gauge theory on a spin manifold with 2-cocycle boundary, i.e., $\partial\Sigma_i = 2\gamma_i, \forall i = 1, 2, \dots, L$. Hence the flux quantization of double covering leads to a form [18, 24, 25]

$$\begin{aligned} & \exp(2\pi i \int_{\Sigma_i} (q_1 F + gG)) \\ &= (-1)^{\omega(\Sigma_i)} \exp(2\pi i \oint_{\gamma_i} 2(q_1 A + ga)), \forall i = 1, 2, \dots, L \end{aligned} \quad (21)$$

where $\omega(\Sigma_i)$ is a second Stiefel-Whitney class of Σ_i . It follows that we can obtain $q_1 = \frac{e}{2N_1^c}$ from the even, i.e., 2-cycle, flux quantization at the boundary on a 4D manifold with spin structures. This remarkable result reveals that the system can have degenerate ground states on closed topologically nontrivial space-time-reversal-protected gapless surface states that are characterized by $(-1)^{\omega(\Sigma)}$ at the boundary. In a TRS TI, Z_2 topological objects of fractionalized charge can emerge from the 2-cocycle flux quantization at the boundary.

It has been known that an effective FT can have a very serious problem of TRS broken spontaneously because of strongly interacting nonabelian $SU(N_1^c)$ [27]. In order to resolve the broken TRS problem, we have focused on a more fundamental concept and deep insight concerning Dirac quantization conditions over a Riemannian manifold with spin structures of topological property. These spin structures can naturally provide the second Stiefel-Whitney characteristic classes for the conditions of flux quantization in the non-empty boundary manifold with spin structures. Since the gauge bundle $SU(N_1^c)$ can naturally be connected to a basis manifold, we have exploited a mathematical method to lift it to a total covering space that unifies the gauge bundle manifold into the basis one. This lifting scheme can lead to the quantization condition on a strongly correlated nonabelian gauge theory

without having the seriously broken TRS. Therefore, under the quantization conditions of complex spinor fields, we can construct an effective topological QFT for studying a nonabelian gauge field theory of strongly correlated electrons on the Riemannian manifold with nonempty boundary spin manifold in correlated TIs.

IV. SPIN CHERN-SIMON THEORY AND INDEX THEOREM OF EFFECTIVE FIELD THEORY

Thus far we have investigated the Dirac flux quantization with the total $U(1) \times SU(N_f^c)$ quark field on the complex spinor representations of antisymmetric parton wavefunctions. The interactions yield the quarks to condense at low energies into a noninteracting T-invariant TI state with axion angle θ . Using Dirac flux quantization, index theorem, and spin CS theory, i.e., half-integer CS theory, we manipulate an effective topological QFT for the total $U(1) \times SU(N_f^c)$ quark field strength $q_1 F + gG$ on the complex spinor representations of antisymmetric parton wavefunctions. It is assumed that M_4 has spin structures with a 2-cycle boundary of double covering. In particular, we should include the above results of the 2-cycle boundary quantization into the topological term of the FT. In the topological term $\frac{\theta}{2\pi} \frac{e^2}{2\pi} \mathbf{E} \cdot \mathbf{B} = \frac{\theta e^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ for noninteracting TIs, the $U(1)$ electron field strength is replaced by $U(1) \times SU(N_f^c)$. Let us construct a partition function

$$\mathcal{Z} = C(-1)^\omega \exp\left(i \int_{M_4} d^3x dt \mathcal{L}_{eff}(F, G)\right). \quad (22)$$

An effective Lagrangian for the gauge theory of total quarks is expressed by $\mathcal{L}_{eff} = \mathcal{L}_0 + \mathcal{L}_{top}$ where $\mathcal{L}_0 = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu}$ is the kinetic Yang-Mills Lagrangian. The second topological term of \mathcal{L}_{eff} has a following form

$$\begin{aligned} \mathcal{L}_{top} &= \frac{\theta_1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr}[(q_1 F_{\mu\nu} + g G_{\mu\nu})(q_1 F_{\rho\sigma} + g G_{\rho\sigma})] \\ &= \partial_\mu \epsilon^{\mu\nu\rho\sigma} \left\{ \frac{\theta_{eff} e^2}{8\pi^2} A_\nu \partial_\rho A_\sigma + \frac{\theta_1 g^2}{8\pi^2} (a_\nu \partial_\rho a_\sigma + \mathcal{L}_{\nu\rho\sigma}) \right\} \end{aligned} \quad (23)$$

where Tr denotes the trace in the N_f^c representation of $SU(N_f^c)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + ig[a_\mu, a_\nu]$ indicate $U(1)$ and $SU(N_f^c)$ field strengths, respectively, and θ_1 is an action angle for $N_f = 1$. Here we denote that $\theta_{eff} = \frac{\theta_1}{2N_f^c}$ and $\mathcal{L}_{\nu\rho\sigma} = i\frac{2}{3} g a_\nu a_\rho a_\sigma$. It is also noted that crossed terms such as $\text{Tr}(F_{\mu\nu} f_{\rho\sigma})$ vanishes owing to the tracelessness of the $SU(N_f^c)$ gauge field. Using spin CS theory, we rewrite the partition function

$$\begin{aligned} \mathcal{Z} &= C(-1)^\omega \exp\left[i \int_{M_4} d^3x dt \mathcal{L}_0 + i\theta_1 g^2 CS^s(a)\right] \\ &\times \exp i \int_{M_3} \left\{ \frac{\theta_{eff} e^2}{8\pi^2} A \wedge dA + \frac{\theta_1 g^2}{8\pi^2} \mathcal{L} \right\}, \quad (24) \end{aligned}$$

where $CS^s(a)$ is a spin CS action defined by $\frac{1}{8\pi^2} \int_{M_3} a \wedge da$ on a 3D manifold with spin structure M_3 which is a boundary of M_4 .

Concerning the spin CS action, Jenquin developed spin CS theory over the space of connections on the basis of the theorem of Dai and Freed regarding the ξ -invariant of the Dirac operator [28, 29]. Exploiting these results, the spin CS actions and the ξ -invariant can be identified up to multiplication with a metric dependent function $f(h_a)$ which is given by [30]

$$e^{2\pi i CS^s(a)} = e^{i\pi \xi(D_a)} f(h_a), \quad (25)$$

where D_a is a Dirac operator on a compact 3D manifold M_3 , and h_a is a metric on it. The ξ -invariant is described by $\xi(D_a) = \frac{1}{2}(\eta_{D_a}(0) + \dim \ker D_a)$. Here $\eta_{D_a}(0)$ refers to the η -invariant of $D_a(0)$ defined by $\eta_{D_a}(s) = \sum_{\lambda \neq 0} \frac{\text{sign} \lambda}{|\lambda|^s}$, $\text{Re}(s) > \frac{3}{2}$. In 3D, it follows that $\xi(D_a) \bmod 2$ is also a smooth function of the geometric parameters.

In order to understand the singular behavior of $\xi(D_a)$, let us consider a singular Dirac monopole in a strongly correlated topological insulator. The singular Dirac is a classical solution of Maxwell's equations that becomes invariant under the translations and rotations of \vec{r} , and takes singularity at the line defined by $\vec{r} = 0$. Assume that $G_{U(1)}$ is a curvature of a $U(1)$ connection a . Then, Maxwell's equations can be solved on $\mathbb{R}^3 \setminus 0$, i.e., on the complement of the point $\vec{r} = 0$ in \mathbb{R}^3 , by $G_{U(1)} = \frac{i}{2} * d(\frac{1}{|\vec{r}|})$ where $*$ is a Hodge operator that explicitly gives the dependence on the orientation of the normal bundle to the line. If Y is a two-sphere enclosing the singularity around $\vec{r} = 0$, then $\frac{i}{2\pi} \int_Y G_{U(1)} = 1$. The $U(1)$ field allows us to consider dyons that are bound states of electric and magnetic charges. In particular, the Witten effect of TIs produces charge $\frac{\theta}{2\pi} \frac{eg_{min}}{2\pi} e$ to the monopole where the minimal strength is written by $g_{min} = \frac{4\pi}{e}$ [34]. The singular part of $G_{U(1)}$ enables us to take $G_{U(1)} \approx \frac{ine}{4} * d(\frac{1}{|\vec{r}|})$ due to the θ term of the Witten effect. It follows that there can be two topological objects with fractional charge $ne/2$ in the strongly correlated TI. The TI ground state has a gap to all topological excitations, and preserves the TRS. It should take ground state degeneracy (GSD) on topologically non-trivial spaces with non-contractible loops corresponding to arbitrary cycles. The ground states can be represented by configurations $(\{\gamma_i, n_i\})$ of Z_2 flux through the non-contractible loops $\gamma_i, \forall i = 1, 2, \dots, N$. Therefore, these topological objects can be described by the representation of a holonomy $\text{Hol}_a(\{\gamma_i, n_i\})$ of a around any loop γ_i . Furthermore the $(-1)^\omega$ factor of Eq. (24) can be replaced by this holonomy.

Accounting for the holonomy of the topological objects, and substituting Eq. (25) into Eq. (24), the parti-

tion function yields [30]

$$\mathcal{Z} = C(-1)^{\frac{\xi(D_a)g^2\theta_1}{2\pi}} \text{Hol}_a(\{\gamma_i, n_i\}) \cdot K(h_a, \theta, g) \times \exp(i \int_{M_4} d^3x dt \mathcal{L}_0 + i \frac{\theta_{eff} e^2}{8\pi^2} \int_{M_3} A \wedge dA), \quad (26)$$

where $K(h_a, \theta, g)$ denotes

$$K(h_a, \theta, g) = [f(h_a) \exp(-\frac{2}{3}g \int_{M_3} a \wedge a \wedge a)]^{\frac{\theta_1 \lambda_1^c}{2\pi N_1^c}},$$

with $\lambda_1^c \equiv N_1^c g^2$ fixed. As $N_1^c \rightarrow \infty$ under the fixed λ_1^c , $\frac{\theta_1 \lambda_1^c}{2\pi N_1^c} \rightarrow 0$. Thus it is shown that $K(h_a, \theta, g)$ goes to 1. These facts give rise to the partition function in a simple form

$$\mathcal{Z} \approx C(-1)^{\frac{\xi(D_a)g^2\theta_1}{2\pi}} \text{Hol}_a(\{\gamma_i, n_i\}) \times \exp(i \int_{M_4} d^3x dt \mathcal{L}_0 + i \frac{\theta_{eff} e^2}{8\pi^2} \int_{M_3} A \wedge dA). \quad (27)$$

V. FRACTIONAL MAGNETOELECTRIC EFFECTS AND FRACTIONAL SURFACE QUANTUM HALL EFFECT

One can study fractional magnetoelectric effects and surface quantum Hall effect by means of the partition function described by Eq. (27) on the effective topological QFT in strongly correlated TIs. The electromagnetic response allows one to calculate the effective axion angles θ_{eff} . The T-invariance provides one for quantization of θ_{eff} in integer multiples of π provided that the minimal electric charge becomes an effective one e_{eff} owing to the holonomy and under the minimum integer case of $n_i = 1$ in Eq. (27). For θ_1 , one can make the following choices using the current parton model of quarks in the absence of breaking the TRS. It follows that there can be two kinds of choices such that $\frac{\xi(D_a)g^2\theta_1}{2\pi} = 2k_1$ or $2k_1 + 1$ for any k_1 . Thus for the choice of the lowest value of $k_1 = 0$ in θ_1 , the effective axion angles are given by two expressions

$$\theta_{eff} = \frac{\theta_1}{2N_1^c} = 0, \quad \frac{2\pi}{\xi(D_a)\lambda_1^c}. \quad (28)$$

Let us generalize the N_1^c case to the effective field theory for multiple flavor values $N_f^c \geq 1$. Assume that quarks of flavor f produces a noninteracting TI with $\theta_f = \frac{2\pi(2k_f+1)}{\xi(D_a)g^2}, \forall k_f$. After integrating them, the effective theory has a gauge group of $U(1) \times \prod_{f=1}^{N_f} U(N_f^c)/U_e(1)$ where $U(1)_e$ indicates the overall $U(1)$ gauge transformation of the electron operator. This gauge group yields the electromagnetic axion angle $\theta_{eff} = \{\sum_{f=1}^{N_f} \frac{N_f^c}{\theta_f}\}^{-1}$. If θ_f is considered to be $\theta_f = \frac{2\pi}{\xi(D_a)g^2}$ for each flavor,

then there can be degenerate axion angles of even integers such as $\theta_{eff} = \frac{2\pi}{\xi(D_a)} \sum_{f=1}^{N_f} \frac{1}{2\lambda_f^c}$ where $\lambda_f^c \equiv N_f^c g^2$. Thus, the effective axion angle yields a simple expression

$$\theta_{eff} = \frac{2\pi}{\xi(D_a)(2g^2 N(N_f, c))} \equiv \frac{\pi}{\xi(D_a)\lambda(N_f, c)}, \quad (29)$$

where $N(N_f, c)$ indicates $N(N_f, c) = N_1^c + \dots + N_{N_f}^c$ as an odd number, and $\lambda(N_f, c) \equiv g^2 N(N_f, c)$.

The partition function of the gauge fields enables one to determine important physical properties for the interacting TI. In general, the surface of the interacting TI has an action domain wall with the electromagnetic action angle which jumps from θ_{eff} in the fractional TI to 0 in the vacuum. Hence by flux quantization, the domain wall of interacting TI leads to the surface QHE

$$\sigma_H^s = \frac{\theta_{eff}}{2\pi} \frac{e^2}{h} = \frac{1}{2\xi(D_a)\lambda(N_f, c)} \frac{e^2}{h}. \quad (30)$$

If $\xi(D_a) = 2$ in 3D, then the surface QHE gives rise to a simple form

$$\sigma_H^s = \frac{1}{4\lambda} \frac{e^2}{h}. \quad (31)$$

In the viewpoint of the magnetoelectric response for the interacting TIs, the fractional axion angle allows one to result in the fractional magnetoelectric polarization $K_\theta = \frac{1}{4\lambda}$. It is proposed that the fractional magnetoelectric effect can be realized in gapped time reversal invariant topological insulators of strongly correlated bosons or fermions with an effective axion angle $\theta_{eff} = \frac{\pi}{2\lambda}$ if they can have fractional excitations and degenerate ground states on topologically nontrivial and oriented spaces.

On the other hand, provided that an effective charge can be expressed in terms of $e_{eff} = \frac{e}{\sqrt{2\lambda}}$, the surface Hall conductance is regarded as a conductivity $\sigma_H^s = \frac{e_{eff}^2}{2h}$ for a single Dirac cone of gapless fermions with fractional charge e_{eff} and a pure axion angle $\theta = \pi$. Recently, angular photoemission spectroscopy experiment has showed that there is a topological phase transition from a trivial insulator into a topological surface state by displaying the emergence of spin vortex with fractional charge $\pm \frac{e}{2}$ and $\theta = \pi$ in a tunable topological insulator BiTe(Si_{1- δ} Se _{δ}) [19]. Theoretically it has been suggested that there can be a quantized vortex of fractional charge $\pm \frac{e}{2}$ and an odd number of gapless Dirac fermions at the surface of a strong TI [20]. In the viewpoint of these experimental and theoretical studies, if $\lambda(N_f, c) = 2$ as well as $\xi(D_a) = 2$, then one can obtain a well-known formula for the surface Hall conductivity of massless Dirac fermions

$$\sigma_H^s = \frac{\tilde{e}^2}{2h}, \quad (32)$$

where $\tilde{e} = \frac{e}{2}$ as a fractional charge. This theoretical construction shows that the TIs can have fractionalized

charge $\frac{e}{2}$ topological objects with a bulk gap and string-like vortex excitations. These topological excitations can be described in terms of a deconfined Z_2 gauge theory in (3+1)D. Due to topologically protected gapless surface states, the unexpected result above has been emerged from the bulk θ term on a new topological phase of quantum matter in (3 + 1)D. If the surface of strongly correlated TIs is covered by time-reversal breaking materials, the bulk θ term enables us to observe the surface Hall conductivity of Eq. (32).

So far, in a fundamental view point, we have focused a topological invariant on the boundary of the strongly correlated TIs. The topological invariance provides us for some interesting physical properties without knowing the microscopic Hamiltonian of the strongly correlated systems. In this article, one of the main focuses is to study the topological invariance on the boundary of the strongly correlated TIs with nonabelian gauge theories without taking the microscopic Hamiltonian of TIs in details. Under the topological invariance, the effective action can be written as a full function of $g^2 N(N_f, c)$ [31, 32]

$$S_{eff} = - \sum_{g=0}^{\infty} N(N_f, c)^{2-2g} F_g(g^2 N(N_f, c)) \quad (33)$$

where the sum over topologies becomes explicit. In Eq. (33), the 't Hooft limit gets to be $N(N_f, c) \rightarrow \infty$ and a 't Hooft coupling $\lambda = g^2 N(N_f, c)$, fixed [31, 32]. $F_g(g^2 N(N_f, c))$ is a scaling function of λ . Thus we can represent the effective action of $g^2 N(N_f, c)$ as the fixed value of λ , and as a sum over surface topology.

Let us consider a surface Hall conductance on an oriented surface with the genus in topology. Then the key idea of topological consideration is to investigate the surface Hall conductance of topological objects on the surface of the genus g in correlated TIs with spin structures. There can be geometric cycles for all even codimensions in general complex structures on a given surface. Todd defined polynomials in these cycles which can represent for a variety of dimension a certain number called the Todd genus [33]. The Todd genus can be expressed in terms of the Chern classes of tangent bundles on the fundamental cycle as the Euler characteristic of a differentiable manifold.

To measure the GSD on topologically nontrivial spatial 3-manifolds, consider a fractional TI on a manifold $\Sigma_g \times I$ where Σ_g is a Riemann surface of Todd genus g and $I = [0, t]$ stands for a bounded interval when t is the sample thickness, and two copies of Σ_g are two bounding surfaces at each ends of I . A noninteracting TI with a $\nu = \frac{1}{2}$ Laughlin state deposited on both surfaces is described by two independent CS theories [30] and has a ground state of 2^g on each surface for a total GSD of $(2^g)^2 = 2^{2g}$ under the Todd genus. This is a disjoint union of the tori which is called the Picard group. In the $g - 1$ component of the

Picard group, the spin line bundles are represented by 2^{2g} points. Due to the Atiyah-Patodi-Singer index theorem [36], the variation of the ξ -invariant under the change of smooth parameters gives rise to $\xi(D_a, t) - \xi(D_a, 0) = \int_{M_3 \times [0, t]} [\hat{A}(\frac{R(s)}{2\pi}) \wedge \text{ch}(G_a(s))]$ where \hat{A} is the roof-genus as the Todd class of an almost complex manifold. $R(s)$ is the Riemann tensor corresponding to the metric $h(s)$ and $\text{ch}(G_a(s))$ is a Chern class [30]. Therefore it is shown that there can exist a genuine fractional TI such that $K_\theta = \frac{1}{2\lambda(N_f, c)(\xi(D_a, t) - \xi(D_a, 0))}$ and Hall conductance

$$\sigma_H^s = \frac{1}{2\lambda(N_f, c)(\xi(D_a, t) - \xi(D_a, 0))} \frac{e^2}{h}. \quad (34)$$

VI. SUMMARY AND CONCLUSION

Having exploited flux quantization, index theorem and spin CS theory, we construct an effective topological field theory of strongly correlated topological insulators coupling to a nonabelian gauge field $SU(N)$ with an interaction constant g in the absence of the TRS breaking. If N and g allow us to define a 't Hooft parameter λ as $\lambda = Ng^2$, then our construction leads to FQHE on the surface with Hall conductance $\sigma_H^s = \frac{1}{4\lambda} \frac{e^2}{2h}$. In particular, if $\lambda = 2$, then it has been shown that $\sigma_H^s = \frac{e^2}{2h}$. For the magnetoelectric response described by a bulk axion angle θ , it has been proposed that the fractional magnetoelectric effect can be realized in gapped time reversal invariant topological insulators of strongly correlated bosons or fermions with $\theta = \frac{\pi}{2\lambda}$ if they can have fractional excitations and degenerate ground states on topologically nontrivial and oriented spaces. Thus there can exist degenerate ground states of FQHE on the surface with Hall conductance of Eqs. (31), (32) and (34). Under our current effective field theory, it is claimed that the topological band structure from the nontrivial Hamiltonian matrix may provide the nontrivial properties of the quantum states with fractionalization and the emergent nonabelian gauge theory without the microscopic models of strongly correlated TIs. As one of future works, it would be very interesting to apply the current theory to the confinement-deconfinement issue in topological quantum chromodynamics by the $SU(N)$ nonabelian gauge field.

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